Synchronization-based approach for estimating all model parameters of chaotic systems

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The problem of dynamic estimation of all parameters of a model representing chaotic and hyperchaotic systems using information from a scalar measured output is solved. The variational calculus based method is robust in the presence of noise, enables online estimation of the parameters and is also able to rapidly track changes in operating parameters of the experimental system. The method is demonstrated using the Lorenz, Rossler chaos, and hyperchaos models. Its possible application in decoding communications using chaos is discussed.

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Synchronization of unidirectionally coupled chaotic systems has been a subject of great interest for over a decade [1–4]. The interest in understanding the synchronization characteristics of chaotic systems stems from its potential applications in a variety of areas, e.g., in optics, communications, and time series analysis of chaotic systems. An important issue in time series analysis of chaotic systems is the estimation of all parameters using information from a scalar measured output. This information can in turn be used to estimate all the unmeasured system states, provided the model is known.

In general, there are three key issues in parameter estimation of dynamical systems. First, the method has to be robust in the presence of noise. Second, the method must allow estimation of all parameters using any conveniently measurable output from the system. Third, it must be able to rapidly track changes in the operating parameters of the experimental system.

Current parameter estimation techniques can be broadly classified as online and off-line strategies. The online, e.g., adaptive control approach [5], though simple to implement, has been demonstrated to be unsuitable for estimation of multiple parameters (e.g., for the Rossler system). In contrast, off-line, e.g., autosynchronization [6] and error minimization [7,8] schemes have been demonstrated to be able to estimate all parameters. The former is a geometric approach where the optimal vector fields governing temporal evolution of the parameters are obtained using a linearization based numerical procedure. Using the error minimization approach, Goodwin *et al.* have shown that all parameters can be estimated by using a scalar measurable output (not necessarily corresponding to one of the state space variables) of chaotic and hyperchaotic systems [8].

For systems with multiple parameters, the least squares error function possesses several minima [8]. This can lead to an erroneous estimate of parameters owing to convergence at one of the local minima. Moreover, in practice, it may not be always possible to possess information of the time at which parameter changes in the experimental system occur. For online schemes, a lack of this information results in erroneous estimates of the determined parameters. Off-line schemes cannot be used in this situation, since there is no way to incorporate the effect of parameter changes in the least squares minimization method traditionally employed for parameter estimation.

In this paper, the least squares approach is used to develop a general and robust method for deriving the dynamical system governing the evolution of all parameters of a chaotic system. The technique is demonstrated using simulated experimental data from the Lorenz, Rossler chaos, and Rossler hyperchaos systems. The advantages of the method are its ability to (a) estimate all parameters in an online setting, (b) respond to unknown changes in parameters of the experimental system, and (c) converge to the "true" estimates of the parameters. Each of these is a distinct improvement over capabilities of the existing methods.

We begin by briefly describing the set up of the parameter estimation problem. Let

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\mathbf{p}) \tag{1}$$

represent the experimental dynamical system with state variables $\mathbf{x} \in \mathbf{R}^n$, whose parameters $\mathbf{p} \in \mathbf{R}^m$ are to be estimated. The overdot indicates differentiation with respect to time t. The only information available from this experimental system is (i) the functional form of the model and (ii) a scalar time series given by an observable $s(\mathbf{x})$. The model is given by the following equation:

$$\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y},\mathbf{q}), \tag{2}$$

where $\mathbf{y} \in \mathbf{R}^n$ and $\mathbf{q} \in \mathbf{R}^m$, and the functional form of $\mathbf{g}$ is identical to that of $\mathbf{f}$ in Eq. (1). *We assume that there exists a unidirectional coupling scheme using the available scalar output of the experimental system (1) which enables asymptotic synchronization of the model system (2) with the experimental system (1)*, i.e., $\mathbf{y} \to \mathbf{x}$ as $t \to \infty$, *if* $\mathbf{q} = \mathbf{p}$. The coupling can either be a drive–response coupling scheme, e.g., the Pecora-Carroll scheme [1] or a feedback coupling scheme [9]. The conditions which enable synchronization of the model system (2) with the experimental system (1) are well known. In most cases, synchronization is guaranteed if all conditional Lyapunov exponents of the error system $\dot{\mathbf{e}} = \dot{\mathbf{x}} - \dot{\mathbf{y}}$ constructed using Eqs. (1) and (2) are negative [10]. Techniques are now available which enable design of an unidirectional scheme which guarantees synchronization [11].

For the sake of conciseness, we only consider feedback coupling in this report. The general representation of this scheme is given by

$$\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y},\mathbf{q}) - \mathbf{B}\mathbf{K}^T[s(\mathbf{y}) - s(\mathbf{x})], \tag{3}$$

where **B** is a constant vector and **K** is the gain vector. The scheme (3) has been previously used for achieving identical synchronization in a hyperchaotic system [9].

We now develop the main theme of this paper which can be defined as "Develop a formalism for constructing a system of differential equations governing the evolution of the model system parameters **q** such that $(\mathbf{y},\mathbf{q}) \to (\mathbf{x},\mathbf{p})$ as $t \to \infty$." Our objective will be to design a parameter evolution scheme that asymptotically drives the measured error $s(\mathbf{y}) - s(\mathbf{x})$ to zero, and thereby yields $(\mathbf{y},\mathbf{q}) \to (\mathbf{x},\mathbf{p})$ as $t \to \infty$. The starting point is the following minimization problem:

$$G(\mathbf{q}) = \min[\{s(\mathbf{y}) - s(\mathbf{x})\}^2]. \tag{4}$$

We note that an inability to correctly estimate the initial conditions of the state variables and/or parameters could result in large errors during the initial stage of evolution of the model system. Our goal, reflected in the choice of the cost function (4), is to force the model output $s(\mathbf{y})$ to asymptotically synchronize with the experimental output $s(\mathbf{x})$. The minimization problem (4) is studied as the following equivalent system of differential equations

$$\dot{q}_j = -\frac{\partial G}{\partial q_j} = -2[s(\mathbf{y}) - s(\mathbf{x})]\frac{\partial s(\mathbf{y})}{\partial q_j}, \quad j = 1, \ldots, m. \tag{5}$$

The equilibrium state of the system (5) is typically attained when the synchronization condition is satisfied, i.e., when $s(\mathbf{y}) = s(\mathbf{x})$. This ensures that the parameter estimates attain their true values. In contrast, convergence to the true parameters is not guaranteed in the error minimization approach since the least squares cost function has several local minima [8]. A knowledge of the variational derivatives $\partial y_i / \partial q_j$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$ is needed for solving this system of equations. Since the functional form of the model is known, these derivatives are given by [using Eq. (3)]

$$\frac{d}{dt}\left(\frac{\partial y_i}{\partial q_j}\right) = \sum_{k=1}^{n} \frac{\partial g_i}{\partial y_k}\frac{\partial y_k}{\partial q_j} + \frac{\partial g_i}{\partial q_j} - \mathbf{B}\mathbf{K}^T \sum_{k=1}^{n} \frac{\partial s(\mathbf{y})}{\partial q_j},$$
$$i = 1, \ldots, n; \quad j = 1, \ldots, m. \tag{6}$$

Formulation of Eqs. (4) and (5) and using Eq. (6) to solve Eq. (5) are the key steps in the proposed procedure for estimating all parameters of a chaotic or hyperchaotic system. The method consists of solving (i) the experimental system (1) (when real experimental data is not available), (ii) the model system (3), (iii) the equations

$$\dot{q}_j = -\epsilon_j \frac{\partial G}{\partial q_j} = -2\epsilon_j[s(\mathbf{y}) - s(\mathbf{x})]\frac{\partial s(\mathbf{y})}{\partial q_j}, \quad j = 1, \ldots, m \tag{7}$$

governing evolution of the parameters, and (iv) the equations corresponding to the evolution of the variational derivatives Eq. (6). The vector of additional parameters $\boldsymbol{\epsilon}$ is needed for guaranteeing stability of the overall system and it also controls the rate of synchronization. When the actual scalar measured output from an experiment is available, an extended system comprising of $n+m+nm$ equations needs to be solved in order to estimate m parameters of a n-dimensional system. The condition for convergence of the procedure is that the real part of the eigenvalues of the Jacobian matrix or the conditional Lyapunov exponents of the extended system formed using Eqs. (3) and (7) are all less than zero [6].

Our first example is the Lorenz system. We demonstrate the method when the scalar observable is x_2. The simulated experimental and model systems are, respectively, given by the following equations:

$$\dot{x}_1 = p_1(x_2 - x_1), \quad \dot{y}_1 = q_1(y_2 - y_1),$$
$$\dot{x}_2 = p_2 x_1 - x_2 - x_1 x_3, \quad \dot{y}_2 = q_2 y_1 - y_2 - y_1 y_3 - k(y_2 - x_2), \tag{8}$$
$$\dot{x}_3 = x_1 x_2 - p_3 x_3 \quad \dot{y}_3 = y_1 y_2 - q_3 y_3,$$

with $p_1 = 10$, $p_2 = 28$, and $p_3 = 8/3$. The six equations (8), together with four equations [Eqs. (6) and (7)] for estimation of each of the three parameters results in a set of eighteen equations, governing the evolution of the (i) simulated experimental system, (ii) model system, (iii) parameters, and (iv) the variational derivatives. When $\mathbf{p} = \mathbf{q}$, the feedback term corresponding to the product of the constant vector $\mathbf{B} = [0,1,0]^T$, suitable gain vector $\mathbf{K} = [0,k,0]^T$, and the output error $(y_2 - x_2)$ guarantees synchronization of the model system with the experimental system. We consider the situation where additive uniformly distributed random noise in the range $[-0.5, 0.5]$ is present in the measured output x_2. Figure 1 shows the evolution of the parameters and the relative estimation errors for the case where in addition to noise, a step perturbation in parameters is imposed on the simulated experimental system. *It is important to note that this introduces an additional complexity since information about the imposed perturbation is not available to the model*. The robustness of the method is demonstrated by the convergence to the original parameters close to $t = 125$; followed by a rapid, stable transition, and subsequent convergence into the vicinity of the new operating parameters. All parameters could be successfully determined when the measured output was the x_1 variable. However, the method fails when the measured output is the x_3 variable, since the Lyapunov exponents of Eq. (3) are not negative for any choice of **B** and **K**.

Our next example is the Rossler system and we demonstrate the method when the scalar observable is x_2. The simulated experimental and model system equations are given by

$$\dot{x}_1 = -x_2 - x_3, \quad \dot{y}_1 = -y_2 - y_3,$$
$$\dot{x}_2 = x_1 + p_1 x_2, \quad \dot{y}_2 = y_1 + q_1 y_2 - k(y_2 - x_2), \tag{9}$$
$$\dot{x}_3 = p_2 + x_3(x_1 - p_3), \quad \dot{y}_3 = q_2 + y_3(y_1 - q_3),$$

with $p_1 = 0.2$, $p_2 = 0.2$, and $p_3 = 9$. The feedback parameter vectors were selected to be $\mathbf{B} = [0,1,0]^T$ and $\mathbf{K} = [0,k,0]^T$. Results for two different cases are shown in Fig. 2. These

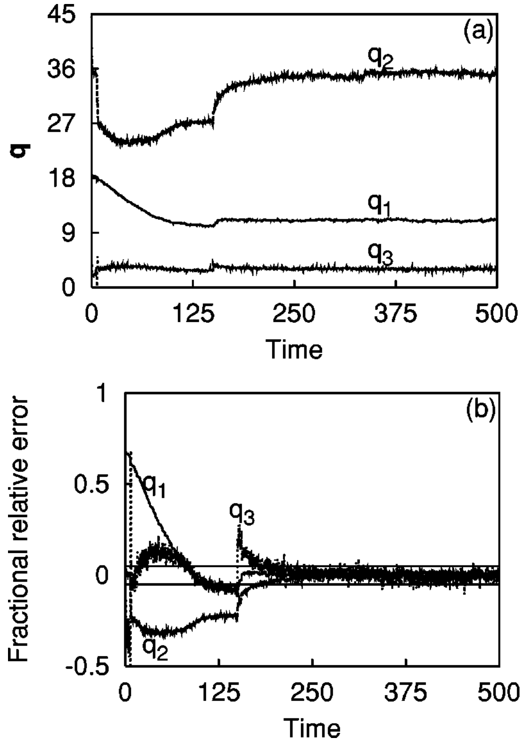

FIG. 1. Temporal evolution of (a) all the three parameters of the Lorenz system, (b) fractional relative errors for the case when additive noise is present in the measured output and the parameters of the simulated experimental system are changed to $p_1=11$, $p_2=35$, and $p_3=3$ at $t=150$. The stability parameters are $k=25$, $\epsilon_1=1$, $\epsilon_2=15$, and $\epsilon_3=1$. The straight lines in (b) correspond to an error of $\pm5\%$.

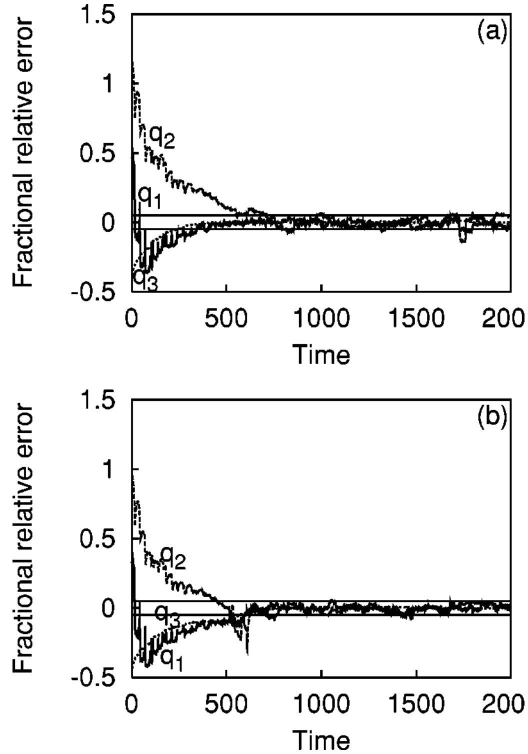

FIG. 2. Temporal evolution of the estimation errors for all the parameters of the Rossler system (9) for the case when (a) only additive noise is present in the measured output x_2, (b) additive noise is present and each of the parameters of the simulated experimental system is increased by 10% at $t=500$. The parameter q_3 has been scaled down 50 times for greater clarity of representation. The stability parameters are $k=20$, $\epsilon_1=0.15$, $\epsilon_2=0.2$, and $\epsilon_3=2$. The straight lines correspond to an error of $\pm5\%$.

correspond to the situation when (i) additive uniformly distributed random noise in the range $[-0.1,0.1]$ is present in the measured output [Fig. 2(a)] and (ii) in addition to noise, the parameters of the simulated experimental system are increased by 10% at $t=500$ [Fig. 2(b)]. For both cases, the parameter estimates exhibit small-amplitude fluctuations around the correct value. The nature of evolution of errors in Fig. 2(b) indicates that the step perturbation results in a rapid transition of the parameter estimates into the vicinity of the new value. A feature of this result is the direct dependence of the magnitude of fluctuations of individual parameters on the manner in which the parameter is related to the measured output.

The next example is a four parameter Rossler hyperchaos system. The simulated experimental and model systems are given by the following equations:

$$\begin{aligned}
\dot{x}_1&=-x_2-x_3, \quad \dot{y}_1=-y_2-y_3,\\
\dot{x}_2&=x_1+p_1x_2+x_4, \quad \dot{y}_2=y_1+q_1y_2+y_4-w,\\
\dot{x}_3&=p_2+x_1x_3, \quad \dot{y}_3=q_2+y_1y_3,\\
\dot{x}_4&=-p_3x_3+p_4x_4, \quad \dot{y}_4=-q_3y_3+q_4y_4-w,
\end{aligned} \tag{10}$$

with $p_1=0.25$, $p_2=3$, $p_3=0.5$, and $p_4=0.05$. The measurable output is assumed to be $s(\mathbf{x})=x_2+x_4$, $w=k(s(\mathbf{y})-s(\mathbf{x}))$, $\mathbf{B}=[0,1,0,1]^T$, and $\mathbf{K}=[0,k,0,k]^T$. It is possible to achieve identical synchronization for this system using unidirectional coupling and a suitable choice of $\mathbf{K}$. We study the ability of the method to track changes in operating parameters of the simulated experimental system. It is assumed that parameters p_1 and p_2 of the simulated experimental system are increased by 4% at $t=500$. The scaled temporal evolution of all the estimated parameters $q_i, i=1,\ldots,4$ is shown in Fig. 3(a). Figure 3(b) shows the evolution of the estimated parameters for the case when in addition to changes in operating parameters of the simulated experimental system, uniformly distributed random noise in the range $[-0.005,0.005]$ is present in the experimental measured output. For each case, it can be seen that the method allows a rapid convergence to the new operating parameters.

Finally, we present a simple example to illustrate a possible application of the proposed parameter estimation scheme in communications using chaos. The problem relates to decoding of an encoded message signal. It is assumed that the following information is known: (i) the chaotic system used to encode the message and (ii) the state variable used for encoding the message. For simplicity, the message is taken to be a sinusoidal function, and it is assumed that the

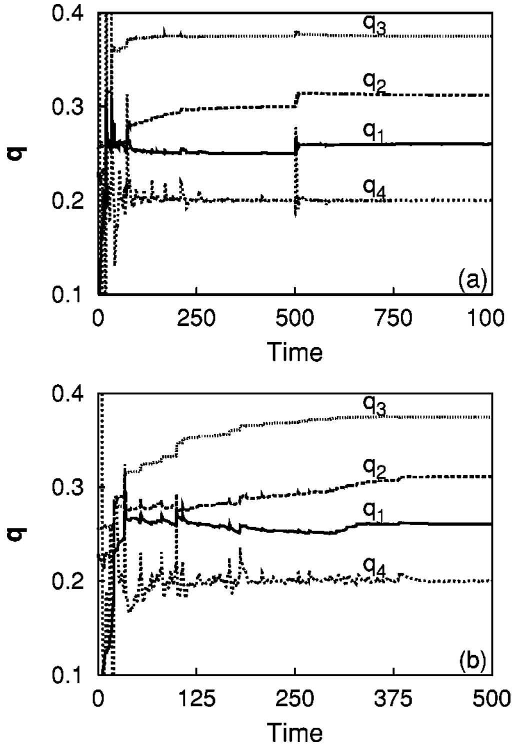

FIG. 3. Temporal evolution of the four parameters q_1, q_2, q_3, and q_4 of the Rossler hyperchaos system (10) for the case when (a) the parameters p_1 and p_2 are increased by 4% at $t=500$, (b) additive noise is present in the measured output x_2+x_4, and the parameters p_1 and p_2 of the simulated experimental system are increased by 4% at $t=300$. For greater clarity of representation, the parameters have been scaled in the following manner: q_2 and q_3 are scaled down by a factor of 10 and $\frac{3}{4}$, respectively, while the parameter q_4 is scaled up by a factor of 4. The stability parameters are (a) $k=4$, $\epsilon_1=0.75$, and $\epsilon_2=\epsilon_3=\epsilon_4=0.005$; (b) $k=3.5$, $\epsilon_1=0.80$, and $\epsilon_2=\epsilon_3=\epsilon_4=0.002$.

encoding is carried out in an additive manner. The objective is to decode the noisy transmitted signal and retrieve the message. The first step in achieving this objective is to estimate the parameters of the model system. Once this has been accomplished, the message can be retrieved by subtracting

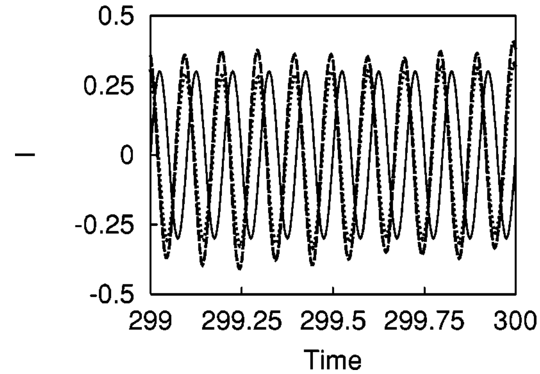

FIG. 4. Comparison of the message signal $I=\sin(20\pi t)$ and the decoded signals obtained using the x_2 variable of Lorenz and Rossler models as the measured output. These are represented by the continuous, dashed and dotted lines, respectively. Decoding results in phase shift as well as in a reduction of the amplitude. The amplitude of the message signal has been scaled for greater clarity. The stability parameters are the same as that given in Fig. 1 and Fig. 2.

the computed model output from the transmitted signal. Figure 4 shows the fair degree of comparison between the message signal and the decoded signal estimated using the Lorenz and Rossler systems for the case when additive noise in the range $[-0.1,0.1]$ is also present in the transmitted signal.

In conclusion, we have developed an analytical framework for the robust design of dynamical systems that guarantees online estimation of all model parameters of a given chaotic-hyperchaotic system. A possible application in communications using chaos has been demonstrated. Past results indicate that the method proposed here would be applicable to the more realistic situation where only discrete-time measurements of the experimental output are available [8].

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